Algebra and Proof - Questions by Topic		
	Binomial Theorem	
1	Express the binomial expansion of $(a^2-3)^4$	**4**
2	Write down and simplify the general term in the expansion of $\left(x^2+\frac{1}{x}\right)^{10}$ Hence or otherwise obtain the term in x^{14}	**3** **2**
3	Show that $\binom{n+1}{3}-\binom{n}{3}=\frac{1}{2}n(n-1)$	**4**
	Complex Numbers	
4	Given the equation $z+2i\bar{z}=8+7i$ express z in the form $a+ib$	**4**
5	Given $z=1+2i$ express $z^2(z+3)$ in the form $a+ib$ Hence or otherwise verify that $1+2i$ is a root of the equation $z^3+3z^2-5z+25=0$ Obtain the other roots of this equation	**2** **2** **2**
6	Express $z=\ -i+\frac{1}{1-i}$ in the form $x+iy$ stating the values for x and y. Find the modulus and the argument of z and plot these on an Argand diagram.	**7**
7	Identify the locus in the complex plane given by $\lvert z+i\rvert=2$	**3**

	Mathematical Proof	
8	Let n be a natural number. For each of the following statements, decide whether it is true or false. If true, give a proof, if false give a counterexample **A** If n is a multiple of 9 then so is n^2 **B** If n^2 is a multiple of 9 then so is n	**4**
9	Let n be an integer. Using proof by contrapositive, show that if n^2 is even, then n is even	**4**
10	Prove by induction that $\frac{d^n}{dx^n}(xe^x) = (x+n)e^x$ for all integers $n \geq 1$	**5**
11	Prove by induction that, for all positive integers n, $$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$ Hence state the limit of $$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$$	**5**

	Matrices	
12	Calculate the inverse of the matrix $\begin{pmatrix} 2 & x \\ -1 & 3 \end{pmatrix}$. For what value of x is this matrix singular	**4**
13	Find the values(s) of p for which the matrix $A = \begin{pmatrix} p & 2 & 0 \\ 3 & p & 1 \\ 0 & -1 & -1 \end{pmatrix}$ is singular.	**4**
14	Given the matrix $A = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$ Show that $A^2 + A = kI$ for some constant k Obtain the values of p and q for which $A^{-1} = pA + qI$	**4** **2**
15	Use Gaussian elimination to solve the system of equations $$\begin{gathered} x + y + 3z = 2 \\ 2x + y + z = 2 \\ 3x + 2y + 5z = 5 \end{gathered}$$	**5**
16	Use Gaussian elimination to solve the system of equations below when $\lambda \neq 2$ $$\begin{gathered} x + y + 2z = 1 \\ 2x + \lambda y + z = 0 \\ 3x + 3y + 9z = 5 \end{gathered}$$	**6**
	Number Theory	
17	Use the Euclidean Algorithm to show that $(231,17) = 1$ where (a, b) denotes the highest common factors of a and b. Hence find integers x and y such that $231x + 17y = 1$	**4**
18	Use the division algorithm to express 1323_{10} in base 7	**3**

	Sequences and Series	
19	The first term of an arithmetic sequence is 2 and the twentieth term is 97 Obtain the sum of the first 50 terms.	**4**
20	The sum $S(n)$, of the first n terms of a sequence $u_1, u_2, u_3, \ \ . \ . \ .$ is given by $S(n) = 8n - n^2, \ \ n \ \geq 1$ (a) Calculate the values of u_1, u_2, u_3 and state what type of sequence it is. (b) Obtain a formula for u_n in terms of n, simplifying your answer	**3** **2**
21	The first and fourth terms of a geometric series are 2048 and 256 respectively Calculate the value of the common ratio. Given that the sum of the first n terms is 4088 find the value of n	**5**
22	The second and third terms of a geometric series are -6 and 3 respectively. State why the associated geometric series has a sum to infinity and obtain this sum.	**5**
23	Given that $u_k = 11 - 2k, (k \geq 1)$. Obtain a formula for $$S_n = \sum_{r=1}^{n} u_k$$ Find the values of n for which $S_n = 21$	**3** **2**

Algebra and Proof - Answers		
	Binomial Theorem	
1	$$(a^2)^4+4(a^2)^3(-3)+6(a^2)^2(-3)^2+4(a^2)(-3)^3+(-3)^4$$ $$=a^8-12a^6+54a^4-108a^2+81$$	**4**
2	The general term is $$\binom{10}{r}(x^2)^{10-r}\left(\frac{1}{x}\right)^r=\binom{10}{r}x^{20-2r}x^{-r}=\binom{10}{r}x^{20-3r}$$ For x^{14} $\quad 20-3r=14$ so $r=2$ $$\binom{10}{2}x^{14}=\frac{10!}{2!\,8!}x^{14}=\frac{10\cdot 9}{2!}x^{14}=\mathbf{45x^{14}}$$	**3** **2**
3	$$\binom{n+1}{3}-\binom{n}{3}=\frac{(n+1)!}{3!\,(n+1-3)!}-\frac{n!}{3!\,(n-3)!}=\frac{(\boldsymbol{n}+\mathbf{1})!}{\mathbf{3}!\,(\boldsymbol{n}-\mathbf{2})!}-\frac{\boldsymbol{n}!}{\mathbf{3}!\,(\boldsymbol{n}-\mathbf{3})!}$$ $$=\frac{(n+1)n\,(n-1)(n-2)!}{3!\,(n-2)!}-\frac{n(n-1)(n-2)(n-3)!}{3!\,(n-3)!}$$ $$=\frac{(n+1)n(n-1)}{3!}-\frac{(n)(n-1)(n-2)}{3!}$$ $$=\frac{n}{3!}[(n+1)(n-1)-(n-1)(n-2)]$$ $$=\frac{n}{6}[n^2-1-(n^2-3n+2)]$$ $$=\frac{n}{6}(3n-3)=\frac{1}{2}\mathrm{n(n-1)}\quad \text{as required}$$	**4**
	Complex Numbers	
4	$$z=a+ib,\quad \bar{z}=a-ib$$ $$LHS\quad z+2i\bar{z}=a+ib+2i(a-ib)$$ $$=a+ib+2ai-2bi^2$$ $$=(a+2b)+(2a+b)i$$ $$(a+2b)+(2a+b)i=8+7i$$ $$a+2b=8,\qquad 2a+b=7,\quad a=2 \text{ and } b=3,\quad \boldsymbol{z}=\mathbf{2}+\mathbf{3}\boldsymbol{i}$$	**4**

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$$z^2(z+3) = (1+2\text{i})(1+2\text{i})(4+2\text{i}) = (1+2i)(10i) = -20+10i$$

$$z^3+3z^2-5z+25 = (1+2i)^3+3(1+2i)^2-5(1+2i)+25$$

$$= -11-2i-9+12i-5-10i+25 = 0$$

No remainder so $z = 1+2i$ is a root of the equation

If $z = 1+2i$ is a root, then $z = 1-2i$ is also a root

$$(z-1-2i)(z-1+2i) = z^2-2z+5$$

$$\rightarrow z^3+3z^3-5z+25 \div z^2-2z+5 = z+5$$

Hence the roots are $\boldsymbol{z = 1 \pm 2i}$ and $\boldsymbol{z = -5}$

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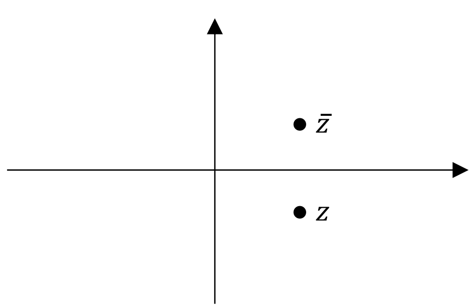
$$-i+\frac{1}{1-i} = -i+\frac{1}{1-i}\times\frac{1+i}{1+i} = -i+\frac{1+i}{2} = \frac{-2i+1+i}{2} = \frac{1}{2}-\frac{1}{2}i$$

$$x = \frac{1}{2}, \qquad y = -\frac{1}{2},$$



$$|z| = \sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \boldsymbol{\frac{\sqrt{2}}{2}} \qquad \arg(z) = \tan^{-1}(-1) = \left(\boldsymbol{\frac{3\pi}{4}}\right) \boldsymbol{or\ 135°}$$

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$$|z+i| = 2$$

$$|x+iy+i| = |2|$$

$$|x+i(y+1)| = 2$$

$$\boldsymbol{x^2+(y+1)^2 = 2^2}$$

This a circle with a centre $(0,-1)$and a radius of 2 units

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	Mathematical Proof	
8	$suppose\ \ m = 9n\ \ \ then\ \ m^2 = (9n)^2 = 81n^2 = 9(9n^2)$ This m^2 is a multiple of 9 and so **A** is true $6^2 = 36 = 9(4)$, but 6 is not divisible by 9 so **B** is not true	4
9	**Contrapositive statement If $\mathbf{n}$ is odd, then $\mathbf{n}^2$ is odd** *An odd number takes the form* $n = 2k + 1$ $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ Since $2(2k^2 + 2k) + 1$ is odd, then the contrapositive statement is true and therefore the original statement is true	4
10	For $n = 1$, LHS $\frac{d}{dx}(xe^x) = e^x + xe^x = (1+x)e^x = (x+1)e^x$ RHS $(x+1)e^x$ so true for $n = 1$ Assume that the statement is true for $n = k$ $$\frac{d^k}{dx^k}(xe^x) = (x+k)e^x$$ Now consider $n = k + 1$ $$\frac{d^{k-1}}{dx^{k+1}}(xe^x) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(xe^x)\right)$$ $$= \frac{d}{dx}(x+k)e^x$$ $$= 1(e^x) + e^x(x+k)$$ $$= e^x(1+x+k) = e^x\big(x + (k+1)\big)$$ So statement is true for $n = k + 1$. Since true for $n = 1, n = k$ and $n = k + 1$, then by induction this is true for all positive integers n	5

11	For $n = 1$, LHS $\frac{1}{1\times2\times3} = \frac{1}{6}$ RHS $\frac{1}{4} - \frac{1}{2\times2\times3} = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$ so true for $n = 1$ Assume that the statement is true for $n = k$ $$\sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(k+1)(k+2)}$$ Now consider $n = k+1$ $$\sum_{r=1}^{k+1} \frac{1}{r(r+1)(r+2)} = \sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)} + f(k+1)$$ $$= \frac{1}{4} - \frac{1}{2(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$ $$= \frac{1}{4} + \frac{2-(k+3)}{2(k+1)(k+2)(k+3)}$$ $$= \frac{1}{4} - \frac{k+1}{2(k+1)(k+2)(k+3)}$$ $$= \frac{1}{4} - \frac{1}{2(k+2)(k+3)}$$ $$= \frac{1}{4} - \frac{1}{2\big((k+1)+1\big)\big((k+2)+1\big)}$$ So statement is true for $n = k+1$. Since true for $n = 1, n = k$ and $n = k+1$, then by induction this is true for all positive integers n	**5**
	Matrices	
12	$$Inverse = \frac{1}{6+x}\begin{pmatrix} 3 & -x \\ 1 & 2 \end{pmatrix}, \quad 6 + x = 0, \ \boldsymbol{x = -6}$$	**4**
13	For a singular matrix $detA = 0$ $$\det A = \mathrm{p} \times \det\begin{pmatrix} p & 1 \\ -1 & -1 \end{pmatrix} - 2 \times \det\begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix} + 0 \times \det\begin{pmatrix} 3 & p \\ 0 & -1 \end{pmatrix}$$ $$\det A = p(-p-(-1)) - 2(-3-0) + 0(-3-0) = 0$$ $$0 = -p^2 + p + 6, \quad (3-p)(p+2) = 0, \quad \boldsymbol{p = 3}, \ \boldsymbol{p = -2}$$	**4**

14	$A^2+A=\begin{pmatrix}0&4&2\\1&0&1\\-1&-2&-3\end{pmatrix}\begin{pmatrix}0&4&2\\1&0&1\\-1&-2&-3\end{pmatrix}+\begin{pmatrix}0&4&2\\1&0&1\\-1&-2&-3\end{pmatrix}$ $=\begin{pmatrix}2&-4&-2\\-1&2&-1\\1&2&5\end{pmatrix}+\begin{pmatrix}0&4&2\\1&0&1\\-1&-2&-3\end{pmatrix}$ $\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}=2I$ $A^2+A=2I$, therefore $A^{-1}(A^2+A)=A^{-1}2I$ $A+I\ =2A^{-1}$ $\boldsymbol{A^{-1}=\frac{1}{2}A+\frac{1}{2}I}$	**4** **2**
15	$\begin{pmatrix}1&1&3:&2\\2&1&1:&2\\3&2&5:&5\end{pmatrix}$ $\begin{pmatrix}1&1&3:&2\\0&1&5:&2\\0&1&4:&1\end{pmatrix}$ $2R_1-R_2, 3R_1-R_3,$ $\begin{pmatrix}1&1&3:&2\\0&1&5:&2\\0&0&1:&1\end{pmatrix}$ $R_2-R_3,$ $\boldsymbol{z=1},\quad \boldsymbol{y=-3},\quad \boldsymbol{x=2}$	**5**
16	$\begin{pmatrix}1&1&2:&1\\2&\lambda&1:&0\\3&3&9:&5\end{pmatrix}$ $\begin{pmatrix}1&1&2:&1\\0&\lambda-2&-3:&-2\\0&0&3:&2\end{pmatrix}$ $R_2-2R_1, R_3-3R_1,$ $\boldsymbol{z=\frac{2}{3}},\quad (\lambda-2)y-2=-2,\quad \boldsymbol{y=0},\quad x=1-\frac{4}{3}=\boldsymbol{-\frac{1}{3}}$ When $\boldsymbol{\lambda=2}$ the second and third rows of the last matrix are the same which would give an infinite number of solutions.	**6**

	Number Theory	
17	$$231 = 17 \times 13 + 10$$ $$17 = 10 \times 1 + 7$$ $$10 = 7 \times 1 + 3$$ $$7 = 3 \times 2 + 1$$ $$3 = 1 \times 3 + 0$$ Hence $(231,17) = 1$ $$1 = 7 - 3 \times 2$$ $$1 = 7 - 2 \times (10 - 7 \times 1) = -2 \times 10 + 3 \times 7$$ $$1 = -2 \times 10 + 3 \times (17 - 10 \times 1) = 3 \times 17 - 5 \times 10$$ $$1 = 3 \times 17 - 5 \times (231 - 13 \times 17) = 68 \times 17 - 5 \times 231$$ $$\boldsymbol{x = -5}, \quad \boldsymbol{y = 68}$$	**4**
18	$$1323 = 189 \times 7 + 0$$ $$189 = 27 \times 7 + 0$$ $$27 = 3 \times 7 + 6$$ $$3 = 0 \times 7 + 3$$ $$1323_{10} = \boldsymbol{3600_7}$$	**3**
	Sequences and Series	
19	$$U_{20} = 2 + d(19) \rightarrow \quad 97 = 2 + 19d, \ \ d = 5$$ $$S_{50} = \frac{1}{2}50(4 + 49 \times 5) = \boldsymbol{6225}$$	**4**
20	$$s(1) = 7, \ \ s(2) = 12, \ \ S(3) = \ 15$$ $$u_3 = 3, \ \ u_2 = 5, u_1 = 7$$ This is an arithmetic sequence where $a = 7$ and the common difference is -2 $$u_n = a + d(n-1) = 7 - 2(n-1) = \boldsymbol{9 - 2n}$$	**3** **2**

21	$U_1 = ar^0 \rightarrow \ a = 2048, \qquad U_4 = ar^3 \rightarrow \ \ 256 = 2048r^3$ $r^3 = \sqrt[3]{\frac{256}{2048}} = \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$ $4088 = \frac{2048\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$ $\frac{4088}{2\times 2048} = 1 - \left(\frac{1}{2}\right)^n$ $\left(\frac{1}{2}\right)^n = \frac{1}{512}, \qquad 2^n = 512, \qquad n = \log_2 512 \ , \boldsymbol{n = 9}$	**4**
22	$U_2 = ar^1 \rightarrow \ -6 = ar^1, \quad U_3 = ar^2 \rightarrow \ \ 3 = ar^2$ $\frac{-6}{r} = \frac{3}{r^2}, \quad r = -\frac{1}{2}\ , \ a = 12$ Sum to infinity exists when $\lvert r \rvert < 1, \ \ -1 < -\frac{1}{2} < 1$ meets this condition $S_\infty = \frac{12}{1+\frac{1}{2}} = \mathbf{8}$	**5**
23	$S_n = \sum_{r=1}^{n} 11 - 2k = \sum_{r=1}^{n} 11 + 2\sum_{r=1}^{k} k$ $= 11n - 2\left(\frac{n}{2}(n+1)\right)$ $= 11n - n^2 - n \ = \ 10n - n^2$ $10n - n^2 = 21, \ \ n^2 - 10n + 21 = 0, \qquad (n-3)(n-7) = 0$ For $S_n = 21 \ \ \boldsymbol{n = 3 \ \text{ and } 7}$	**3** **2**